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Solutions Of Certain Systems Of Linear Differential Equations

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SOLUTIONS OF CERTAIN SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

By

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L.A.W., Sr.

DEDICATION

To my wife, Verna Mae Ware, without whose help this paper could not have been possible.

To my children, Barbara, Lionel, Jr., Vera, and Larry, may this accomplishment prove to be incentive toward greater heights of learning.

L.A.W. Sr.

SYSTEMS OF SIMULTANEOUS LINEAR DIFFERENTIAL EQUATIONS

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CHAPTER I

INTRODUCTION AND TERMINOLOGY

This thesis is the result of an extension of an investigative paper submitted in partial fulfillment of the requirements for the Degree of Bachelor of Science in Mathematics by Raymond L. Scott, May, 1956. Scott discussed systems of linear differential equations with constant coefficients containing four equations with four variables. We shall extend this theory to systems of n equations in n variables.

Very often in applied mathematics, physics, chemistry, one is faced with the problem of solving systems of linear differential equations. Finding solutions of systems of linear differential equations often proved to be very difficult. Many methods have been developed in connection with this problem. A rather general method for solving systems of ordinary linear differential equations with constant coefficients with developed in this paper.

The problem will be discussed in the following manner:

Chapter II will be devoted to developing the theory relative to the nature and the operation with linear differential operators.

Chapter III will be devoted to developing the theory of the operators method relative to general systems of ordinary linear differential equations with constant coefficients in a certain manner.

Chapter IV will be devoted to applications to above methods to special systems.

Chapter V will be devoted to a summary of the results of this investigation of the above stated problem.

Certain special terminology will be used throughout this paper. Therefore, this will be stated presently.

Definition 1. $D = \frac{d}{dx}$ or $Du = \frac{du}{dx}$ where u is any function of x will be called a linear differential operator.[1]

Definition 2.

$$(D - m)y = f(x)$$

$$y = \frac{1}{D - m} f(x) = (D - m)^{-1} f(x) \qquad (1)$$

The function $(D-m)^{-1}$ is called an inverse operator.

Definition 3.

$$P(D) = A_0 D^n + A_1 D^{n-1} + A_2 D^{n-2} + \dots + A_{n-1} D + A_n$$

is defined to be a polynomial of degree n in the operator D. The operator D has certain very important properties. They are; namely:

Property 1.

$$D^k(cf) = cD^k f \qquad \text{(Commutation)}$$

Property 2.

$$D^n(u + v) = D^n u + D^n v \qquad \text{(Distribution)}$$

Property 3.

$$P(D) = (D - m_1)(D - m_2) \dots (D - m_n) \qquad \text{(Factorization)}$$

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[1] Lyman M. Kells, Elementary Differential Equations, (New York: McGraw-Hill Book Company, Inc., 1954), p. 74.

CHAPTER II

LINEAR DIFFERENTIAL OPERATORS

The operators discussed in this chapter furnish powerful and time-saving methods of solving many differential equations of outstanding importance. Here the usefulness of operators is clarified by the following theorems:

THEOREM I.

H_1 : Let $P(D) = D - m_1$.

H_2 : Let $P(D)y = f(x)$ or $y = \frac{1}{P(D)} f(x)$.

Then

$$C: \quad y = e^{m_1 x}\int_{x_0}^{x} e^{-m_1 x} f(x)dx.$$

Proof:

Let $(D-m)y = f(x)$

$Dy-my = f(x)$ is lineqr and seems to be a first order linear differential equation. Since

$e^{\int Pdx}$ is the integrating factor, we have

$$e^{\int -mdx} = e^{-m\int dx} = e^{-mx}$$

$$e^{-mx} y = \int_{x_0}^{x} e^{-mx} f(x)dx$$

multiplying both sides by e^{mk}

$$y = e^{mx}\int_{x_0}^{x} e^{-mx} f(x)dx. \qquad (2)$$

THEOREM II.

H_1: Let $P(D) = (D - m_1)(D - m_2)(D - m_3) \dots (D - m_n)$.

H_2: Let the m's be distinct.

H_3: Let $P(D)y = f(x)$ or $y = \frac{1}{P(D)} f(x)$.

Then

C: $$y = A_1 e^{m_1 x}\int_{x_0}^{x} e^{-m_1 x} f(x)dx + A_2 e^{m_2 x}\int_{x_0}^{x} e^{-m_2 x} f(x)dx +$$

$$+ A_3 e^{m_3 x}\int_{x_0}^{x} e^{-m_3 x} f(x)dx + \dots + A_n e^{m_n x}\int_{x_0}^{x} e^{-m_n x} f(x)dx.$$

Proof: Let $P(D) = a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_{n-1} D + a_n =$

$$= (D - m_1)(D - m_2)(D - m_3) \dots (D - m_n).$$

If $P(D)y = f(x)$,

then $(D - m_1)(D - m_2)(D - m_3) \dots (D - m_n)y = f(x)$

$$y = \frac{1}{(D - m_1)(D - m_2)(D - m_3) \dots (D - m_n)} f(x).$$

Then by the method of partial fraction, we have

$$y = \frac{A_1}{D - m_1} f(x) + \frac{A_2}{D - m_2} f(x) + \frac{A_3}{D - m_3} f(x) + \dots + \frac{A_n}{D - m_n} f(x)$$

$$y = A_1(D - m_1)^{-1} f(x) + A_2(D - m_2)f(x) + A_3(D - m_3)f(x) + \dots$$

$$+ A_n(D - m_n)^{-1} f(x). \qquad (3)$$

Now the application of Theorem I to each term in equation (3)

$$y = A_1 e^{m_1 x}\int_{x_0}^{x} e^{-m_1 x} f(x)dx + A_2 e^{m_2 x}\int_{x_0}^{x} e^{-m_2 x} f(x)dx +$$

$$+ A_3 e^{m_3 x}\int_{x_0}^{x} e^{-m_3 x} f(x)dx + \dots + A_n e^{m_n x}\int_{x_0}^{x} e^{-m_n x} f(x)dx \quad (4^\circ$$

where the value of the A's are determined by the method of undetermined coefficients. Q.E.D.

THEOREM III.

H_1: Let $P(D) = (D - m_1)(D - m_2)\,(D - m_3)\dots(D - m_n) = D - m_1)^n$

H_2: Let the m's be equal.

H_3: Let $y = \frac{1}{P(D)} f(x)$.

Then

C: $$y = \frac{1}{(D - m_1)^n} = (D - m_1)^n f(x) = e^{m_1 x}\int_{x_0}^{x} \frac{(x - x_0)^{n-1}}{(n-1)!} e^{-m_1 x} f(x)dx.$$

Proof:

$P(D)y = f(x)$ and

$(D - m_1)(D - m_2)(D - m_3)\dots(D - m_n)y = f(x)$

operating with each factor separately and treating $m_2, m_3, m_n = m_1$,

then

$$(D - m_2)(D - m_3)\dots(D - m_n)y = \frac{1}{D - m_1} f(x) = e^{m_1 x}\int_{x_0}^{x} e^{-m_1 x} f(x)dx$$

and

$$(D - m_3)(D - m_4) \dots (D - m_n)y = \frac{1}{D - m_2} e^{m_1 x} \int_{x_0}^{x} e^{-m_1 x} f(x)dx$$

$$= e^{m_2} \int_{x_0}^{x} e^{-(m_2 - m_1)x} \int_{x_0}^{x} e^{-m_1 x} f(x)dxdx$$

since $m_2 = m_1$, it follows that

$$= e^{m_1} \int_{x_0}^{x} dx \int_{x_0}^{x} e^{-m_1 x} f(x)dx$$

$$y = e^{m_1} \int_{x_0}^{x} (x - x_0) e^{-m_1 x} f(x)dx$$

Operating with the third factor, we have

$$(D - m_4)(D - m_5) \dots (D - m_n)y = \frac{1}{D - m_3} e^{m_1} \int_{x_0}^{x} (x - x_0) e^{-m_1 x} f(x)dx$$

Since $m_3 = m_4$, it follows that

$$= e^{m_1 x} \int_{x_0}^{x} \int_{x_0}^{x} (x - x_0) e^{-m_1 x} f(x)dxdx$$

$$= e^{m_1 x} \int_{x_0}^{x} \left[\int_{x_0}^{x} (x - x_0)dx \right] e^{-m_1 x} f(x)dx$$

$$y = e^{m_1 x} \int_{x_0}^{x} \frac{(x - x_0)^2}{2!} e^{-m_1 x} f(x)dx$$

Therefore by the induction process, we have

$$y = e^{m_1 x} \int_{x_0}^{x} \frac{(x - x_0)^{n-1}}{(n - 1)!} f(x)\, dx \qquad (5)$$

Q.E.D.

THEOREM IV.

H_1: Let $P(D) = (D - m_1)(D - m_2) \dots (D - m_n)$.

H_2: Let $P(D)y = 0$.

Then

C: $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$.

Proof: Since $P(D)y = 0$,

let $P(D) = a_o D^n + a_1 D^{n-1} + \dots + a_{n-1} D + a_n$ where the a's are constants. Then

$$(a_o D^n + a_1 D^{n-1} + \dots + a_{n-1} D + a_n)y = 0 \qquad (6)$$

Let $(D + a)y = 0$ represent equation (6) and $R + a$ is the representative of $D + a$, then

$$R + a = 0$$

$$R = -a$$

Therefore, the solution of $(D + a)y = 0$ is

$$y = ce^{-ax}$$

This suggest an equation of the form $y = ce^{rx}$ might be a solution of (6). Substituting $y = ce^{rx}$, $\frac{dy}{dx} = cre^{rx}$, we obtain

$$ce^{rx}(a_o r^n + a_1 r^{n-1} + \dots + a_{n-1} r + a_n) = 0.$$

Since ce^{a} cannot be zero, it follows that

$$ce^{rx}(a_o r^n + a_1 r^{n-1} + \dots + a_{n-1} r + a_n) \text{ is an r equation.}$$

This equation will be satisfied if m is a root of the equation. If the factors of P(D) are

$$(D - m_1)(D - m_2)\dots(D - m_n)$$

and if $m_1, m_2, \dots m_n$ are roots of the polynomial in D it follows

that the conditions of the equation are satisfied.

Therefore the equation has a solution

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \dots + c_n e^{m_n x} .$$

THEOREM V.

H_1: Let $P(D)\,y = \dfrac{Q(D)}{H(D)}\,y$, where Q(D) is a polynomial of degree m and H(D) is a polynomial of degree n.

H_2: Let Q(D) and H(D) have no common factors.

Then

C: $P(D)y = S(D)y + R(D)y$, where $R(D) = \dfrac{K(D)}{T(D)}$.

Case I. $m \geq n$.

$$\text{Let } P(D) = \frac{Q(D)}{H(D)} = \frac{a_o D^m + a_1 D^{m-1} + a_2 D^{m-2} + \dots + a_{m-1} D + a_m}{b_o D^n + b_1 D^{n-1} + b_2 D^{n-2} + \dots + b_{n-1} D + b_n}$$

If $m > n$

then $P(D) = S(D)$, where $S(D)$ is a polynomial of degree $m - n$

and $R(D) = \frac{K(D)}{T(D)}$, where $K(D)$ is a lower degree than $T(D)$.

If $m = n$, then $P(D) = S(D)$ is a constant $+ \frac{K(D)}{T(D)}$.

Case II. $m < n$. Let

$$P(D) = \frac{Q(D)}{H(D)} = \frac{a_o D^m + a_1 D^{m-1} + \dots + a_{m-1} D + a_m}{b_o D^n + b_1 D^{n-1} + \dots + b_{n-1} + b_n}$$

If $m < n$, then $P(D) = \frac{K(D)}{T(D)}$, where $K(D)$ is a lower degree than $T(D)$, and is a rational function of negative degree which may be decomposed into sums by partial fraction. Therefore, applying Theorems 1, 2, 3, 4 above to each of these cases, the theorem follows. Q.E.D.

An example of the inverse operators method follows: where $m < n$:

$$\frac{D^3 - D^2 - 5D - 2}{D^4 + 3D^3 + 2D^2} \sin x, \text{ where } x_o = 0,$$

Hence,

$$\frac{D^3 - D^2 - 5D - 2}{D^2\,(D^2 + 3D + 2)} = \frac{A_1}{D^2} + \frac{A_2}{D} + \frac{A_3}{D+2} + \frac{A_4}{D+1}.$$

The value of the A's are obtained by partial fraction.

$A_1 = -1$; $A_2 = -1$; $A_3 = 1$, $A_4 = 1$.

Placing these values in for the A's, we have

$$\frac{D^3 - D^2 - 5D - 2}{D^2(D^2 + 3D + 2)} = \frac{-1}{D^2} + \frac{-1}{D} + \frac{1}{D+2} + \frac{1}{D+1}$$

And

$$\frac{D^3 - D^2 - 5D - 2}{D^2(D^2 - 5D - 2)} \sin x = \left[(-D - 0)^{2} - (D - 0)^{-1} + (D + 2)^{-1} + (D + 1)^{-1}\right] \sin x$$

$$= -e^{0x}\int_0^x xe^{-0x} \sin x\,dx - e^{0x}\int_0^x e^{-0x} \sin x\,dx + e^{-2x}\int_0^x e^{2x} \sin x\,dx + e^{-x}\int_0^x e^{x} \sin x\,dx$$

$$= \Big[x \cos x + \sin x\Big]_0^x + \Big[\cos x\Big]_0^x + e^{-2x}\left[\frac{e^{2x}(2 \sin x - \cos x)}{5}\right]_0^x + e^{-x}\left[\frac{e^{-x}(\sin x - \cos x)}{2}\right]_0^x$$

$=$

$$= x \cos x + \sin x + \cos x - 1 + \frac{2}{5} \sin x - \frac{1}{5} \cos x + \frac{e^{-2x}}{5} + \frac{1}{2} \sin x - \frac{1}{2} \cos x + \frac{e^{-x}}{2}$$

$=$

$$= x \cos x - 1 + \left(1 + \frac{2}{5} + \frac{1}{2}\right) \sin x + \left(1 - \frac{1}{5}\ 1\ \frac{1}{2}\right) \cos x + \frac{e^{-2x}}{5} + \frac{e^{-x}}{2}$$

Therefore solving by the inverse operators method

$$\frac{D^3 - D^2 - 5D - 2}{D^4 + 3D^3 + 2D^2} \sin x = x \cos x - 1 + \frac{19}{10} \sin x + \frac{3}{10} \cos x + \frac{e^{-2x}}{5} + \frac{e^{-x}}{2}$$

CHAPTER III

THE OPERATOR METHOD APPLIED TO A GENERAL SOLUTION

Now the theory which was developed in the preceding chapter will be applied to a general system of n differential equations in n variables. Let us consider the system :

$$\begin{cases} P_{11}(D)x_1 + P_{12}(D)x_2 + P_{13}(D)x_3 + \dots + P_{1n}(D)x_n = f_1(t) \\ P_{21}(D)x_1 + P_{22}(D)x_2 + P_{23}(D)x_3 + \dots + P_{2n}(D)x_n = f_2(t) \\ P_{31}(D)x_1 + P_{32}(D)x_2 + P_{33}(D)x_3 + \dots + P_{3n}(D)x_n = f_3(t) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ P_{n1}(D)x_1 + P_{n2}(D)x_2 + P_{n3}(D)x_3 + \dots + P_{nn}(D)x_n = f_n(t) \end{cases} \tag{7}$$

where $P_{ij}(D)$ are polynomials in the operator D as well as co-factors of the variables.

$$\text{Let } \Delta = \begin{vmatrix} P_{11}(D) & P_{12}(D) & P_{13}(D) & \dots & P_{1n}(D) \\ P_{21}(D) & P_{22}(D) & P_{23}(D) & \dots & P_{2n}(D) \\ P_{31}(D) & P_{32}(D) & P_{33}(D) & \dots & P_{3n}(D) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ P_{n1}(D) & P_{n2}(D) & P_{n3}(D) & \dots & P_{nn}(D) \end{vmatrix} \tag{8}$$

represent the characteristic determinant of the system.

First we assume that the equations are of the first order and the operator polynomials are of degree one. For our system two cases may exist, the system is independent if the determinant of the system is not equal to zero, or it is dependent if the determinant of the system is zero. In the characteristic determinant, $P_{11}(D)$, $P_{12}(D)$, $P_{13}(D)$, and $P_{1n}(D)$ are operators and may be considered as ordinary numbers, then (7) appears as an algebraic system which may be solved Cramer's rule provided $\Delta(D) \neq 0$. The solutions, $x_1(t)$, $x_2(t)$, ... $x_n(t)$, may be represented as follows:

$$x_1 = \frac{\begin{vmatrix} f_1 & P_{12}(D) & P_{13}(D) \dots P_{1n}(D) \\ f_2 & P_{22}(D) & P_{23}(D) \dots P_{2n}(D) \\ f_3 & P_{32}(D) & P_{33}(D) \dots P_{3n}(D) \\ \vdots & \vdots & \vdots \quad\quad \vdots \\ f_n & P_{n2}(D) & P_{n3}(D) \dots P_{nn}(D) \end{vmatrix}}{\begin{vmatrix} P_{11}(D) & P_{12}(D) & P_{13}(D) \dots P_{1n}(D) \\ P_{21}(D) & P_{22}(D) & P_{23}(D) \dots P_{2n}(D) \\ P_{31}(D) & P_{32}(D) & P_{33}(D) \dots P_{3n}(D) \\ \vdots & \vdots & \vdots \quad\quad \vdots \\ P_{n1}(D) & P_{n2}(D) & P_{n3}(D) \dots P_{nn}(D) \end{vmatrix}}; \quad x_2 = \frac{\begin{vmatrix} P_{11}(D) & f_1 & P_{13}(D) \dots P_{1n}(D) \\ P_{21}(D) & f_2 & P_{23}(D) \dots P_{2n}(D) \\ P_{31}(D) & f_3 & P_{33}(D) \dots P_{3n}(D) \\ \vdots & \vdots & \vdots \quad\quad \vdots \\ P_{n1}(D) & f_n & P_{n3}(D) \dots P_{nn}(D) \end{vmatrix}}{\begin{vmatrix} P_{11}(D) & P_{12}(D) & P_{13}(D) \dots P_{1n}(D) \\ P_{21}(D) & P_{22}(D) & P_{23}(D) \dots P_{2n}(D) \\ P_{31}(D) & P_{32}(D) & P_{33}(D) \dots P_{3n}(D) \\ \vdots & \vdots & \vdots \quad\quad \vdots \\ P_{n1}(D) & P_{n2}(D) & P_{n3}(D) \dots P_{nn}(D) \end{vmatrix}}; \qquad (9)$$

$$x_3 = \frac{\begin{vmatrix} P_{11}(D) & P_{12}(D) & f_1 & \dots & P_{1n}(D) \\ P_{21}(D) & P_{22}(D) & f_2 & \dots & P_{2n}(D) \\ P_{31}(D) & P_{32}(D) & f_3 & \dots & P_{3n}(D) \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ P_{n1}(D) & P_{n2}(D) & f_n & \dots & P_{nn}(D) \end{vmatrix}}{\begin{vmatrix} P_{11}(D) & P_{12}(D) & P_{13}(D) & \dots & P_{1n}(D) \\ P_{21}(D) & P_{22}(D) & P_{23}(D) & \dots & P_{2n}(D) \\ P_{31}(D) & P_{32}(D) & P_{33}(D) & \dots & P_{3n}(D) \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ P_{n1}(D) & P_{n2}(D) & P_{n3}(D) & \dots & P_{nn}(D) \end{vmatrix}}; \quad \dots \quad x_n = \frac{\begin{vmatrix} P_{11}(D) & P_{12}(D) & P_{13}(D) & \dots & f_1 \\ P_{21}(D) & P_{22}(D) & P_{23}(D) & \dots & f_2 \\ P_{31}(D) & P_{32}(D) & P_{33}(D) & \dots & f_3 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ P_{n1}(D) & P_{n2}(D) & P_{n3}(D) & \dots & f_n \end{vmatrix}}{\begin{vmatrix} P_{11}(D) & P_{12}(D) & P_{13}(D) & \dots & P_{1n}(D) \\ P_{21}(D) & P_{22}(D) & P_{23}(D) & \dots & P_{2n}(D) \\ P_{31}(D) & P_{32}(D) & P_{33}(D) & \dots & P_{3n}(D) \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ P_{n1}(D) & P_{n2}(D) & P_{n3}(D) & \dots & P_{nn}(D) \end{vmatrix}}$$

where the x_n represents the n^{th} term.

This solution may be written in still another form:

$$x_1(t) = \frac{\Delta P_{11} D}{\Delta(D)} f_1(t) + \frac{\Delta P_{21}(D)}{\Delta(D)} f_2(t) + \frac{\Delta P_{31}(D)}{\Delta(D)} f_3(t) + \dots + \frac{\Delta P_{n1}(D)}{\Delta(D)} f_n(t) \tag{10}$$

$$x_2(t) = \frac{\Delta P_{12}(D)}{\Delta(D)} f_1(t) + \frac{\Delta P_{22}(D)}{\Delta(D)} f_2(t) + \frac{\Delta P_{32}(D)}{\Delta(D)} f_3(t) + \dots + \frac{\Delta P_{n2}(D)}{\quad} f_n(t)$$

$$x_3(t) = \frac{\Delta P_{13}(D)}{\Delta(D)} f_1(t) + \frac{\Delta P_{23}(D)}{\Delta(D)} f_2(t) + \frac{\Delta P_{33}(D)}{\Delta(D)} f_3(t) + \dots + \frac{\Delta P_{n3}(D)}{\Delta(D)} f_n(t)$$

$$\begin{matrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix}$$

$$x_n = \frac{\Delta P_{1n}(D)}{\Delta(D)} f_1(t) + \frac{\Delta P_{2n}(D)}{\Delta(D)} f_2(t) + \frac{\Delta P_{3n}(D)}{\Delta(D)} f_3(t) + \dots + \frac{\Delta P_{nn}(D)}{\Delta(D)} f_n(t)$$

where $\Delta(D)$ is the characteristic determinant (8) and $\Delta P_{11}(D)$, $\Delta P_{21}(D)$, $\Delta P_{31}(D)$, $\Delta P_{n1}(D)$ are cofactors of elements $P_{11}(D)$, $P_{21}(D)$, $P_{31}(D)$, $P_{n1}(D)$ in the determinant $\Delta(D)$. Each of the fractions is a sum integral extended of the integral (t, t_o) from (1) Chapter II.

THEOREM I: If the operator polynomials of a system are all of degree ≤ 1 and if the determinant formed from them is degree n, then the system has one and only one solution satisfying the initial conditions

$$x_1(t) = x_2(t) = x_3(t) = x_n(t) \qquad (\text{ when } t = t_o)$$

which is given by Cramer's rule.[1]

Let

$$\begin{cases} P_{11}(D)x_1 + P_{12}(D)x_2 + P_{13}(D)x_3 + \dots + P_{1n}(D)x_n = f_1(t_1) \\ P_{21}(D)x_1 + P_{22}(D)x_2 + P_{23}(D)x_3 + \dots + P_{2n}(D)x_n = f_2(t_2) \\ P_{31}(D)x_1 + P_{32}(D)x_2 + P_{33}(D)x_3 + \dots + P_{3n}(D)x_n = f_3(t_3) \\ \quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad \vdots \\ P_{n1}(D)x_{n1} + P_{n2}(D)x_{n2} + P_{n3}(D)x_{n3} + \dots + P_{nn}(D)x_n = f_n(t_n) \end{cases} \tag{11}$$

be the system of our equations. If cofactors are multiplied by the

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[1] Raymond L. Scott, "Systems of Simultaneous Linear Differential Equations," (Unpublished Bachelor's Thesis, Prairie View Agricultural and Mechanical College, Prairie View, Texas, 1956), p.16.

leading elements $P_{11}(D), P_{21}(D), P_{31}(D), \ldots P_{n1}(D)$ of elements of the first column of the coefficient determinant and then added, when expanded by column of the new determinant formed, the coefficient of x_1 in the sum is equal to the value of the tererminant, while the coefficients of $x_2, x_3 \ldots\ldots x_n$ are all zeros.[2] The sign of each element when brought into the leading position is obtained by the use of the symbol $(-1)^{n-1}$ to determine the sign:

Proof: Setting up the coefficient determinant of the system and solving using the elements of the first column, we have

$$\begin{vmatrix} P_{11}(D) & P_{12}(D) & P_{13}(D) \ldots & P_{1n}(D) \\ P_{21}(D) & P_{22}(D) & P_{23}(D) \ldots & P_{2n}(D) \\ P_{31}(D) & P_{32}(D) & P_{33}(D) \ldots & P_{3n}(D) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ P_{n1}(D) & P_{n2}(D) & P_{n3}(D) \ldots & P_{nn}(D) \end{vmatrix} = \qquad (12)$$

---

[2] Arnold Dresden, Solid Analytical Geometry and Determinants (New York: John Wiley and Sons, Inc., 1948), p.36.

$$
P_{11}(D)\begin{vmatrix} P_{22}(D) & P_{23}(D) & \dots & P_{2n}(D) \\ P_{32}(D) & P_{33}(D) & \dots & P_{3n}(D) \\ \vdots & \vdots & & \vdots \\ P_{n2}(D) & P_{n3}(D) & \dots & P_{nn}(D) \end{vmatrix} x_1 - P_{21}(D)\begin{vmatrix} P_{12}(D) & P_{13}(D) & \dots & P_{1n}(D) \\ P_{32}(D) & P_{33}(D) & \dots & P_{3n}(D) \\ \vdots & \vdots & & \vdots \\ P_{n2}(D) & P_{n3}(D) & \dots & P_{nn}(D) \end{vmatrix} x_1 \tag{13}
$$

$$
P_{31}(D)\begin{vmatrix} P_{12}(D) & P_{13}(D) & \dots & P_{1n}(D) \\ P_{22}(D) & P_{23}(D) & \dots & P_{2n}(D) \\ \vdots & \vdots & & \vdots \\ P_{n2}(D) & P_{n3}(D) & \dots & P_{nn}(D) \end{vmatrix} x_1 - P_{n1}(D)\begin{vmatrix} P_{12}(D) & P_{13}(D) & \dots & P_{1n}(D) \\ P_{22}(D) & P_{23}(D) & \dots & P_{2n}(D) \\ \vdots & \vdots & & \vdots \\ P_{32}(D) & P_{33}(D) & \dots & P_{3n}(D) \end{vmatrix} x_1 =
$$

$$
\begin{aligned}
P_{11}(D)\Big[&P_{22}(D)P_{33}(D)P_{nn}(D) + P_{32}(D)P_{n3}(D)P_{2n}(D) + P_{n2}(D)P_{3n}(D)P_{23}(D) - \\
&- P_{2n}(D)P_{33}(D)P_{n3}(D) - P_{3n}(D)P_{n3}(D)P_{22}(D) - \\
&- P_{nn}(D)P_{32}(D)P_{23}(D)\Big]x_1 - P_{21}(D)\Big[P_{12}(D)P_{33}(D)P_{nn}(D) + \\
&+ P_{32}(D)P_{n3}(D)P_{1n}(D) + P_{n2}(D)P_{3n}(D)P_{13}(D) - \\
&- P_{1n}(D)P_{33}(D)P_{n2}(D) - P_{3n}(D)P_{n3}(D)P_{12}(D) - \\
&- P_{nn}(D)P_{32}(D)P_{13}(D)\Big]x_1 + P_{31}(D)\Big[P_{12}(D)P_{23}(D)P_{nn}(D) + \\
&+ P_{22}(D)P_{n3}(D)P_{1n}(D) + P_{n2}(D)P_{2n}(D)P_{13}(D) -
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
& - P_{1n}(D)P_{23}(D)P_{n2}(D) - P_{2n}(D)P_{n3}(D)P_{12}(D) \\
& - P_{nn}(D)P_{22}(D)P_{13}(D)\Big] x_1 - P_{n1}(D)\Big[P_{12}(D)P_{23}(D)P_{3n}(D) \\
& + P_{22}(D)P_{33}(D)P_{1n}(D) + P_{32}(D)P_{2n}(D)P_{13}(D) \\
& - P_{1n}(D)P_{23}(D)P_{32}(D) - P_{2n}(D)P_{33}(D)P_{12}(D) \\
& - P_{3n}(D)P_{22}(D)P_{13}(D)\Big] x_1 \; .
\end{aligned}
$$

Factoring out x_1 and completing multiplication the expansion will give:

$$
\begin{aligned}
& \Big(P_{11}(D)P_{22}(D)P_{33}(D)P_{nn}(D) + P_{11}(D)P_{32}(D)P_{n3}(D)P_{2n}(D) \\
& + P_{11}(D)P_{n2}(D)P_{3n}(D)P_{23}(D) - P_{11}(D)P_{2n}(D)P_{33}(D)P_{n2}(D) \\
& - P_{12}(D)P_{3n}(D)P_{n3}(D)P_{22}(D) - P_{11}(D)P_{nn}(D)P_{32}(D)P_{23}(D) \\
& - P_{21}(D)P_{12}(D)P_{33}(D)P_{nn}(D) - P_{21}(D)P_{32}(D)P_{n3}(D)P_{1n}(D) \\
& - P_{21}(D)P_{n2}(D)P_{3n}(D)P_{13}(D) + P_{21}(D)P_{1n}(D)P_{33}(D)P_{n2}(D) \\
& + P_{21}(D)P_{3n}(D)P_{n3}(D)P_{12}(D) + P_{21}(D)P_{n2}(D)P_{32}(D)P_{13}(D) \\
& + P_{31}(D)P_{12}(D)P_{23}(D)P_{nn}(D) + P_{31}(D)P_{22}(D)P_{n3}(D)P_{1n}(D) \\
& + P_{31}(D)P_{n2}(D)P_{2n}(D)P_{13}(D) - P_{31}(D)P_{1n}(D)P_{23}(D)P_{n2}(D) \\
& - P_{31}(D)P_{2n}(D)P_{n3}(D)P_{12}(D) - P_{31}(D)P_{nn}(D)P_{22}(D)P_{13}(D) \; . \; . \\
& - P_{n1}(D)P_{12}(D)P_{23}(D)P_{3n}(D) - P_{n1}(D)P_{22}(D)P_{33}(D)P_{1n}(D) \qquad (15) \\
& - P_{n1}(D)P_{32}(D)P_{2n}(D)P_{13}(D) + P_{n1}(D)P_{1n}(D)P_{23}(D)P_{32}(D)
\end{aligned}
$$

$$+ P_{2n}(D)P_{33}(D)P_{12}(D) + P_{2n}(D)P_{3n}(D)P_{22}(D)P_{13}(D) \,)\, x_1 = D_x$$

Substituting the elements of the second column for the leading elements of the first column in (3) and expanding we have for x_2 :

$$
\begin{aligned}
&(P_{12}(D)P_{22}(D)P_{33}(D)P_{nn}(D) + P_{12}(D)P_{32}(D)P_{n3}(D)P_{2n}(D) \\
&+ P_{12}(D)P_{n2}(D)P_{32}(D)P_{23}(D) - P_{12}(D)P_{2n}(D)P_{33}(D)P_{n2}(D) \\
&- P_{12}(D)P_{3n}(D)P_{n3}(D)P_{22}(D) - P_{12}(D)P_{nn}(D)P_{32}(D)P_{23}(D) \\
&- P_{22}(D)P_{12}(D)P_{33}(D)P_{nn}(D) - P_{22}(D)P_{32}(D)P_{n3}(D)P_{1n}(D) \\
&- P_{22}(D)P_{n2}(D)P_{3n}(D)P_{13}(D) + P_{22}(D)P_{1n}(D)P_{33}(D)P_{n2}(D) \\
&+ P_{22}(D)P_{3n}(D)P_{n3}(D)P_{12}(D) + P_{22}(D)P_{nn}(D)P_{32}(D)P_{13}(D) \\
&+ P_{32}(D)P_{12}(D)P_{23}(D)P_{nn}(D) + P_{32}(D)P_{22}(D)P_{n3}(D)P_{1n}(D) \\
&+ P_{32}(D)P_{n2}(D)P_{2n}(D)P_{13}(D) - P_{32}(D)P_{1n}(D)P_{23}(D)P_{n2}(D) \\
&- P_{32}(D)P_{2n}(D)P_{n3}(D)P_{12}(D)P - P_{32}(D)P_{nn}(D)P_{22}(D)P_{13}(D) \ldots \\
&- P_{n2}(D)P_{12}(D)P_{23}(D)P_{3n}(D) - P_{n2}(D)P_{22}(D)P_{33}(D)P_{1n}(D) \\
&- P_{n2}(D)P_{32}(D)P_{2n}(D)P_{13}(D) + P_{n2}(D)P_{1n}(D)P_{23}(D)P_{32}(D) \\
&+ P_{n2}(D)P_{33}(D)P_{12}(D)P_{2n}(D)P + P_{n2}(D)P_{3n}(D)P_{22}(D)P_{13}(D) \,)\, x_2 = 0
\end{aligned}
$$

We have shown that coefficients for x_2 when added in the expansion equal to zero. Therefore we may conclude that the coefficients for $x_1 \ldots x_n$ will also vanish leaving the value for x_1 as the only value and solution satisfying the initial condition.

To prove the uniqueness of one solution let us assume that y_1, y_2, y_3, and y_n be solutions of our system (7). We will prove that the solution is the same solution satisfying the conditions

$$x_1(t) = x_2(t) = x_3(t) = x_n(t) = 0. \quad (\text{when } t = t_o).$$

If we operate on our equations by multiplying by

$$\frac{\Delta P_{11}(D)}{\Delta(D)}, \quad \frac{\Delta P_{12}(D)}{\Delta(D)}, \quad \frac{\Delta P_{13}(D)}{\Delta(D)}, \quad \frac{\Delta P_{1n}(D)}{\Delta(P)},$$

we have in the solution

$$\sum_{i=1}^{n}\frac{\Delta P_{i1}(D)}{\Delta(D)}\left[P_{i1}(D)y_1\right] + \sum_{i=1}^{n}\frac{\Delta P_{i2}(D)}{\Delta(D)}\left[P_{i2}(D)y_2\right]$$
$$+ \sum_{i=1}^{n}\frac{\Delta P_{i3}(D)}{\Delta(D)}\left[P_{i3}(D)y_3\right] + \dots + \sum_{i=1}^{n}\frac{\Delta P_{in}(D)}{\Delta(D)}\left[P_{in}(D)y_n\right]$$
$$= \sum_{i=1}^{n}\frac{\Delta P_{in}(D)}{\Delta(D)} f_i . \tag{16}$$

Since $P_{i1}(D)$, $P_{i2}(D)$, $P_{i3}(D)$, $P_{in}(D)$ are operator polynominials of degree 1 and the conditions $y_1(t_o) = y_2(t_o) = y_3(t_o) = y_n(t_o) = 0$

are satisfied, the expansions give

$$\begin{aligned} \sum_{i=1}^{n}\frac{\Delta P_{i1}(D)}{\Delta(D)}\left[P_{11}(D)y_1\right] &= \left[\sum_{i=1}^{n}\frac{P_{11}(D)\Delta P_{i1}(D)}{\Delta(D)}\right]y_1 = 1.y_1 \\ \sum_{i=1}^{n}\frac{\Delta P_{i2}(D)}{\Delta(D)}\left[P_{12}(D)y_2\right] &= \left[\sum_{i=1}^{n}\frac{P_{12}(D)\Delta P_{i2}(D)}{\Delta(D)}\right]y_2 = D.y_2 \end{aligned} \tag{17}$$

$$\sum_{i=1}^{n} \frac{\Delta P_{i3}(D)}{\Delta(D)} \left[P_{i3}(D) y_3 \right] = \left[\sum_{i=1}^{n} \frac{P_{i3}(D) \Delta P_{i3}(D)}{\Delta(D)} \right] y_3 = D \cdot y_3$$

$$\sum_{i=1}^{n} \frac{\Delta P_{in}(D)}{\Delta(D)} \left[P_{in}(D) y_n \right] = \left[\sum_{i=1}^{n} \frac{P_{in}(D) \Delta P_{in}(D)}{\Delta(D)} \right] y_n = D \cdot y_n$$

Equations (K) now become

$$y_n = - \sum_{i=1}^{n} \frac{\Delta P_{it}(D)}{\Delta(D)} f_i$$

Now by comparison of the solutions for x_1, x_2, x_3, x_n with solutions for y_1, y_2, y_3, and y_n, our conclusion is apparent that the solutions are one and the same.

THEOREM 2. If the operator polynomials in system (7) are of degree ≤ 2 and if the determinant formed from them is of degree 2n, then system (7) has one and only one solution satisfying the initial conditions.

$x(t) = Dx(t) = y(t) = Dy(t) = z(t) = Dz(t) = 0$ (when $t = t_o$)

and it is given by expressions (10) obtained by formally applying Cramer's rule to system (7).

Proof: Since $\Delta(D)$ is of degree 6 and each of the cofactors Δ

$$\Delta P_{11}(D),\ P_{12}(D),\ P_{13}(D) \ldots P_{1n}(D)$$

is of degree ≤ 4, the rational functions

$$\frac{\Delta P_{11}(D)}{\Delta(D)}, \frac{\Delta P_{12}(D)}{\Delta(D)}, \frac{\Delta P_{13}(D)}{\Delta(D)}, \ldots, \frac{\Delta P_{1n}(D)}{\Delta(D)}$$

are of degree ≤ -2. Hence by rule 5 which states, If $R(q)$ is a rational function of negative degree $-n$, then $R(D)f(x)$ is a function whose derivatives of order $0, 1, 2, \ldots, n-1$ vanish for $x = x_o$.[3] where $y(x) = \frac{1}{P(D)} f(x)$ is a solution of equation $P(D)y = f(x)$ which satisfies the initial condition

$$y(x_o) = y'(x_o) = y''(x_o) = \ldots = y^{n-1}(x_o) = 0,$$

all the terms of the right member of Equation (10) vanish together with their first derivatives, for $t = t_o$. Then the initial conditions are satisfied.

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[3]Michael Golomb and Merrill Shanks, Elements of Ordinary Differential Equations, (New York: McGraw-Hill Book Company, Inc., 1950), p. 198.

CHAPTER IV

APPLICATIONS

In this chapter we will solve systems of ordinary linear differential equations with constant coefficient according to Cramer's rule, as stated in Theorem 1 in Chapter III, using the operator's method.

Example 1.

$$\begin{cases} 4x(t) + 2\dfrac{dy(t)}{dt} = e^t \\ \dfrac{d(x)(t)}{dt} + 2y(t) = 0 \end{cases}$$

The system using operators become

$$\begin{cases} 4x(t) + 2Dy(t) = e^t \\ Dx(t) + 2y(t) = 0 \end{cases}$$

where

$$P_{11}(D) = 4 \; ; \; P_{12}(D) = 2D$$

$$P_{21}(D) = D \; ; \; P_{22}(D) = 2$$

Then

$$\Delta(D) = \begin{vmatrix} 4 & 2D \\ D & 2 \end{vmatrix} = -2(D+2)(D-2)$$

Since $\Delta \neq 0$, then by Cramer's rule

$$x(t) = \frac{\begin{vmatrix} e^t & 2D \\ 0 & 2 \end{vmatrix}}{\begin{vmatrix} 4 & 2D \\ D & 2 \end{vmatrix}} = \frac{2e^t}{-2(D+2)(D-2)} = \frac{1}{(D+2)(D-2)} \cdot e^t$$

$$\frac{1}{(D+2)(D-2)} = \frac{A_1}{D+2} + \frac{A_2}{D-2}$$

Here the A's are found by the method of partial fraction

$$A_1 = -\frac{1}{4} \;;\; A_2 = \frac{1}{4}$$

Then

$$\frac{-1}{(D+2)(D-2)} e^t = -\frac{1}{4}(D+2)^{-1} \cdot e^t + \frac{1}{4}(D-2)^{-1} \cdot e^t$$

$$= -\frac{1}{4} e^{-2t} \int e^{2t} \cdot e^t \, dt + \frac{1}{4} e^{2t} \int e^{-2t} \cdot e^t \, dt$$

$$= -\frac{1}{4} e^{-2t} \int -e^{3t} \, dt - \frac{1}{4} e^{2t} \int e^{-t} \, dt$$

$$= -\frac{1}{4} e^{-2t} \left(-\frac{e^{3t}}{3} + c_1 \right) - \frac{1}{4} e^{2t} \left(-e^{-t} + c_2 \right)$$

$$= \frac{1}{12} e^t - \frac{c_1 e^{-2t}}{4} + \frac{e^t}{4} - \frac{c_2 e^{2t}}{4}$$

$$x(t) = \frac{1}{3} e^t - \frac{c_1 e^{-2t}}{4} - \frac{c_2 e^{2t}}{4}$$

Solving for y(t), we have

$$y(t) = \frac{\begin{vmatrix} 4 & e^t \\ D & 0 \end{vmatrix}}{\begin{vmatrix} 4 & 2D \\ D & 2 \end{vmatrix}} = \frac{-De^t}{-2(D+2)(D-2)} = \frac{1}{(D+2)(D-2)} \cdot \frac{e^t}{2}$$

$$\frac{1}{(D+2)(D-2)} = \frac{A_1}{D+2} + \frac{A_2}{D-2}$$

Here again the A's are found by the method of partial fraction.

$$A_1 = -\frac{1}{4} \; ; \; A_2 = \frac{1}{4}$$

Then

$$\frac{1}{(D+2)(D-2)} \cdot \frac{e^t}{2} = -\frac{1}{4}(D+2)^{-1}\frac{e^t}{2} + \frac{1}{4}(D-2)^{-1}\frac{e^t}{2}$$

$$= -\frac{1}{4}e^{-2t}\int e^{2t}\cdot\frac{e^t}{2}dt + \frac{1}{4}e^{2t}\int e^{-2t}\cdot\frac{e^t}{2}dt$$

$$= -\frac{1}{4}e^{-2t}\left[\frac{1}{2}\int e^{3t}dt\right] + \frac{1}{4}e^{2t}\left[\frac{1}{2}\int e^{-t}dt\right]$$

$$= -\frac{1}{4}e^{-2t}\left(\frac{e^{3t}}{6} + c_1\right) + \frac{1}{4}e^{2t}\left(-\frac{e^{-t}}{2} + c_2\right)$$

$$= -\frac{1}{24}e^t - \frac{c_1 e^{-2t}}{4} - \frac{e^t}{8} + \frac{c_2 e^{2t}}{4}$$

$$y(t) = -\frac{1}{6}e^t - \frac{c_1 e^{-2t}}{4} + \frac{c_2 e^{2t}}{4}$$

Evaluating our solution by substituting the values obtained in the equations of our system, we have

$$\frac{4}{3}e^{t} - \cancel{c_1 e^{-2t}} - \cancel{c_2 e^{2t}} - \frac{1}{3}e^{t} + \cancel{c_1 e^{-2t}} + \cancel{c_2 e^{2t}} = e^{t}$$

$$\frac{4}{3}e^{t} - \frac{1}{3}e^{t} = e^{t}$$

$$e^{t} = e^{t}$$

Substituting in the second equation for x and y, we have

$$\frac{1}{3}\cancel{e^{t}} + \frac{c_1 \cancel{e^{-2t}}}{\cancel{2}} - \frac{c_2 \cancel{e^{2t}}}{\cancel{2}} - \frac{1}{\cancel{3}}e^{t} - \frac{c_1 \cancel{e^{-2t}}}{\cancel{2}} + \frac{c_2 \cancel{e^{2t}}}{\cancel{2}} = 0$$

$$0 = 0$$

Therefore the solutions satisfy the conditions of the system.

Example 2 a. Now let us consider the system

$$\begin{cases} \dfrac{dx(t)}{dt} - x(t) + 2y(t) = 0 \\ 3x(t) + \dfrac{dy(t)}{dt} - 2y(t) = 0 \end{cases}$$

with boundary conditions, when $t = t_0$ $\begin{cases} x(t) = 1 \\ y(t) = 0 \end{cases}$

and a transformation $x(t) = 1 - x(t)$

The system using operators become

$$\begin{cases} (D-1)x(t) + 2y(t) = 0 \\ 3x + (D-2)y(t) = 0 \end{cases}$$

Making the necessary transformation for x(t), we have

$$\begin{cases}(D-1)X + 2y = -1 \\ 3X + (D-2)y = 0\end{cases}$$

$$\Delta(D) = \begin{vmatrix} D-1 & 2 \\ 3 & D-2 \end{vmatrix} = D^2 - 3D - 4 \text{ or } (D-4)(D+1)$$

$$X(t) = \frac{\begin{vmatrix} -1 & 2 \\ 0 & D-2 \end{vmatrix}}{(D-4)(D+1)} = \frac{-D+2}{(D-4)(D+1)}$$

$$\frac{-D+2}{(D-4)(D+1)} = \frac{A_1}{D-4} + \frac{A_2}{D+1}$$

Here the A's are found by the process of partial fraction.

$$A_1 = -\frac{2}{5} \; ; \; A_2 = -\frac{3}{5}$$

$$\frac{-D+2}{(D-4)(D+1)} = -\frac{2}{5}(D-4)^{-1} - \frac{3}{5}(D+1)^{-1}$$

$$= -\frac{2}{5} e^{4t}\int e^{-4t}dt - \frac{3}{5}e^{-t}\int e^{t}dt$$

$$= -\frac{2}{5} e^{4t}\left(-\frac{e^{-4t}}{4} + c_1\right) - \frac{3}{5}e^{-t}\left(e^{t} + c_2\right)$$

$$= \frac{1}{10} - \frac{2}{5}c_1e^{4t} - \frac{3}{5} - \frac{3}{5}c_2e^{-t}$$

$$X(t) = -\frac{1}{2} - \frac{2}{5}c_1e^{4t} - \frac{3}{5}c_2e^{-t}$$

Solving for y(t), we have

$$y(t) = \frac{\begin{vmatrix} D-1 & -1 \\ 3 & 0 \end{vmatrix}}{(D-4)(D+1)} = \frac{3}{(D-4)(D+1)}$$

$$\frac{3}{(D-4)(D+1)} = \frac{A_1}{D-4} + \frac{A_2}{D+1}$$

Here the A'S are determined by the process of partial fraction.

$$A_1 = \frac{3}{5} \;;\; A_2 = -\frac{3}{5}$$

$$\frac{3}{(D-4)(D+1)} = \frac{3}{5}(D-4)^{-1} - \frac{3}{5}(D+1)^{-1}$$

$$= \frac{3}{5}e^{4t}\int e^{-4t}dt - \frac{3}{5}e^{-t}\int e^{t}dt$$

$$= \frac{3}{5}e^{4t}\left(-\frac{e^{-4t}}{4} + c_1\right) - \frac{3}{5}e^{-t}\left(e^{t} + c_2\right)$$

$$= -\frac{3}{20} + \frac{3}{5}c_1e^{4t} - \frac{3}{5} - \frac{3}{5}c_2e^{-t}$$

$$y(t) = -\frac{3}{4} + \frac{3}{5}c_1e^{4t} - \frac{3}{5}c_2e^{-t}$$

The boundary conditions are when $t = t_0$ $\begin{cases} x = 1 \\ y = 0 \end{cases}$

Then $c_1 = -\dfrac{3}{4e^{4t_0}} \;;\; c_2 = -\dfrac{2}{e^{-t_0}}$

For a solution, we have

$$x(t) = -\frac{1}{2} + \frac{3}{10e^{4t_0}}e^{4t} + \frac{6}{5e^{-t_0}}e^{-t}$$

$$y(t) = -\frac{3}{4} - \frac{9}{20e^{4t_0}}e^{4t} + \frac{6}{5e^{-t_0}}e^{-t}$$

Evaluating our solution by substituting the values obtained in the equations of our system. The first equation yields

$$\frac{12}{10\cancel{e^{4t_0}}}e^{4t} - \frac{6}{5\cancel{e^{-t_0}}}e^{-t} + \frac{1}{2} - \frac{3}{10\cancel{e^{4t_0}}}e^{4t} - \frac{6}{5\cancel{e^{-t_0}}}e^{-t} - \frac{3}{2} - \frac{9}{10\cancel{e^{4t_0}}}e^{4t} + \frac{12}{5\cancel{e^{-t_0}}}e^{-t} = -1$$

The second equation yields

$$-\frac{3}{2} + \frac{9}{10\cancel{e^{4t_0}}}e^{4t} + \frac{18}{5\cancel{e^{-t_0}}}e^{-t} - \frac{36}{20\cancel{e^{4t_0}}}e^{4t} - \frac{6}{5\cancel{e^{-t_0}}}e^{-t} + \frac{6}{4} + \frac{18}{20e^{4t_0}}e^{4t} - \frac{12}{5\cancel{e^{-t_0}}}e^{-t} = 0$$

Therefore the solutions satisfy the conditions of the system.

Example 2b.

$$\frac{dx(t)}{dt} - x(t) + 2y(t) = 0$$

$$3x + \frac{dy(t)}{dt} - 2y(t) = 0$$

The boundary condition are , when $t = 0$ $\begin{cases} x(t) = 0 \\ y(t) = 1 \end{cases}$

The system using operators become

$$\begin{cases} (D-1)x + 2y = 0 \\ 3x + (D-2)y = 0 \end{cases}$$

For a transformation, let $y(t) = 1 + Y(t)$, then

$$(D-1)x + 2Y = -2$$

$$3x + (D-2)Y = 2$$

$$\Delta(D) = \begin{vmatrix} D-1 & 2 \\ 3 & D-2 \end{vmatrix} = D^2 - 3D - 4 \text{ or } (D-4)(D+1)$$

$$x(t) = \frac{\begin{vmatrix} -2 & 2 \\ 2 & D-2 \end{vmatrix}}{(D-4)(D+1)} = \frac{-2D}{(D-4)(D+1)}$$

Evaluating $\frac{-2D}{(D-4)(D+1)}$ we have

$$\frac{-2D}{(D-4)(D+1)} = \frac{A_1}{D-4} + \frac{A_2}{D+1}$$

Here the A's are found by the process of partial fraction

$$A_1 = -\frac{8}{5} \,;\; A_2 = -\frac{2}{5}$$

$$\frac{-2D}{(D-4)(D+1)} = -\frac{8}{5}(D-4)^{-1} - \frac{2}{5}(D+1)^{-1}$$

$$= -\frac{8}{5}e^{4t}\int e^{-4t}dt - \frac{2}{5}e^{-t}\int e^{t}dt$$

$$= -\frac{8}{5}e^{4t}\left(-\frac{e^{-4t}}{4} + c_1\right) - \frac{2}{5}e^{-t}\left(e^{t} + c_2\right)$$

$$= \frac{2}{5} - \frac{8}{5}c_1e^{4t} - \frac{2}{5} - \frac{2}{5}c_2e^{-t}$$

$$x(t) = -\frac{8}{5}c_1e^{4t} - \frac{2}{5}c_2e^{-t}$$

Solving for y(t)

$$y(t) = \frac{\begin{vmatrix} D-1 & -2 \\ 3 & 2 \end{vmatrix}}{(D-4)(D+1)} = \frac{2D+4}{(D-4)(D+1)}$$

Evaluating $\frac{2D+4}{(D-4)(D+1)}$ we have

$$\frac{2D+4}{(D-4)(D+1)} = \frac{A_1}{D-4} + \frac{A_2}{D+1}$$

Here the value of the A's are determined by the process of partial fraction.

$$A_1 = \frac{12}{5} \,;\; A_2 = -\frac{2}{5}$$

Then

$$\frac{2D+4}{(D-4)(D+1)} = \frac{12}{5}(D-4)^{-1} - \frac{2}{5}(D+1)^{-1}$$

$$= \frac{12}{5}e^{4t}\int e^{-4t}dt - \frac{2}{5}e^{-t}\int e^{t}dt$$

$$= \frac{12}{5}e^{4t}\left(-\frac{e^{-4t}}{4} + c_1\right) - \frac{2}{5}e^{-t}\left(e^{t} + c_2\right)$$

$$= -\frac{3}{5} + \frac{12}{5}c_1e^{4t} - \frac{2}{5} - \frac{2}{5}c_2e^{-t}$$

$$y(t) = -1 + \frac{12}{5}c_1e^{4t} - \frac{2}{5}c_2e^{-t}$$

The boundary conditions are, when t= 0 $\begin{cases} x = 0 \\ y = 1 . \end{cases}$

Then

$$c_1 = \frac{1}{2}\ ;\ c_2 = -2$$

For a solution, we have

$$x(t) = -\frac{4}{5}e^{4t} + \frac{4}{5}e^{-t}$$

$$y(t) = -1 + \frac{6}{5}e^{4t} + \frac{4}{5}e^{-t}$$

Substituting the values in the first and second equations of our system respectively, we have

$$\text{(a)}\ -\frac{16}{\cancel{5}}e^{4t} - \frac{4}{\cancel{5}}e^{-t} + \frac{4}{\cancel{5}}e^{4t} - \frac{4}{\cancel{5}}e^{-t} - 2 + \frac{12}{\cancel{5}}e^{4t} + \frac{8}{\cancel{5}}e^{-t} = -2$$

$$-2 = -2$$

$$\text{(b)}\ -\frac{12}{\cancel{5}}e^{4t} + \frac{12}{\cancel{5}}e^{-t} + \frac{24}{\cancel{5}}e^{4t} - \frac{4}{\cancel{5}}e^{-t} + 2 - \frac{12}{\cancel{5}}e^{4t} - \frac{8}{\cancel{5}}e^{-t} = 2$$

$$2 = 2$$

Therefore the solutions satisfy the conditions of the system.

Consider the systems of equations of Example III.

Example III.

$$\begin{cases} D x_1^2 + x_2 = 2\cos t \\ (D^2-1)x_2 + D^2 x_3 = 0 \\ (D^2-1)x_1 + D^2 x_3 = 0 \end{cases}$$

Then by Cramer's Rule:

$$x_1 = \frac{\begin{vmatrix} 2\cos t & 1 & 0 \\ 0 & D^2-1 & D^2 \\ 0 & 0 & D^2 \end{vmatrix}}{\begin{vmatrix} D^2 & 1 & 0 \\ 0 & D^2-1 & D^2 \\ D^2-1 & 0 & D^2 \end{vmatrix}} = \frac{\begin{vmatrix} D^2-1 & D^2 \\ 0 & D^2 \end{vmatrix}}{\begin{vmatrix} D^2 & 1 & 0 \\ 0 & D^2-1 & D^2 \\ D^2-1 & 0 & D^2 \end{vmatrix}} = \frac{(D^4-D^2)\,2\cos t}{D^6-D^2} = \frac{D^2-1}{D^4-1}\,2\cos t$$

$$x_2 = \frac{\begin{vmatrix} D & 2\cos t & 0 \\ 0 & 0 & D^2 \\ D^2-1 & 0 & D^2 \end{vmatrix}}{D^2(D^4-1)} = \frac{\begin{vmatrix} 0 & D^2 \\ D^2-1 & D^2 \end{vmatrix}\,(-2\cos t)}{D^2(D^4-1)} = \frac{D^2(D^2-1)}{D^2(D^4-1)} = \frac{D^2-1}{D^4-1}\,2\cos t$$

$$x_3 = \frac{\begin{vmatrix} D & 1 & 2\cos t \\ 0 & D^2-1 & 0 \\ D^2-1 & 0 & 0 \end{vmatrix}}{D^2(D^4-1)} = \frac{\begin{vmatrix} 0 & D^2-1 \\ D^2-1 & 0 \end{vmatrix}\,2\cos t}{D^2(D^4-1)} = \frac{-(D^2-1)^2}{D^2(D^4-1)}\,2\cos t$$

Solving for x_1

$$\frac{D^2-1}{D^4-1} = \frac{(D+1)(D-1)}{(D^2+1)(D+1)(D-1)} = \frac{1}{D^2+1}$$

$$\frac{1}{D^2+1} = \frac{A_1}{D+i} + \frac{A_2}{D-i}$$

Here the A's are found by the method of partial fraction

$$A_1 = \frac{1}{2i} ; \quad A_2 = \frac{1}{2i}$$

Then

$$\frac{1}{D^2+1} 2\cos t = \left[\frac{1}{2i}(D-i)^{-1} + \frac{1}{2i}(D+i)^{-1}\right] 2\cos t$$

$$= -\frac{1}{2i} e^{it} \int_{\tau_0}^{\tau} e^{-it} \cdot 2\cos t\, dt + \frac{1}{2i} e^{-it} \int_{\tau_0}^{\tau} e^{it} \cdot 2\cos t\, dt$$

$$= -\frac{1}{i} e^{it} \int_{\tau_0}^{\tau} e^{-it} \cos t\, dt + \frac{1}{i} e^{-it} \int_{\tau_0}^{\tau} e^{it} \cos t\, dt$$

Let $\cos t = \dfrac{e^{it}+e^{-it}}{2}$, $\quad \sin t = \dfrac{e^{it}-e^{-it}}{2i}$

We have

$$\frac{1}{D^2+1} 2\cos t = -\frac{1}{2i} e^{it} \int_{\tau_0}^{\tau} e^{-it}(e^{it}+e^{-it})\,dt + \frac{1}{2i} e^{-it} \int_{\tau_0}^{\tau} e^{it}(e^{it}+e^{-it})\,dt$$

$$= -\frac{1}{2i} e^{it} \int_{\tau_0}^{\tau} (1+e^{-2it})\,dt + \frac{1}{2i} e^{-it} \int_{\tau_0}^{\tau} (e^{2it}+1)\,dt$$

$$= -\frac{1}{2i} e^{it} \left[t - \frac{e^{-2it}}{2i}\right]_{\tau_0}^{\tau} + \frac{1}{2i} e^{-it} \left[\frac{e^{2it}}{2i} + t\right]_{\tau_0}^{\tau}$$

$$= \left(\frac{te^{-it}}{2i} - \frac{te^{it}}{2i}\right) - \left(\frac{e^{it}}{4} + \frac{e^{it}}{4}\right) + \left(\frac{t_0 e^{i\tau}}{2i} - \frac{t_0 e^{-i\tau}}{2i}\right) + \left(\frac{e^{i(\tau-2\tau_0)}}{4} + \frac{e^{-i(\tau-2\tau_0)}}{4}\right)$$

Factoring out t, t_o, and $\frac{1}{2}$ yields

$$= t\left(\frac{e^{-it}-e^{it}}{2i}\right) - \frac{1}{2}\left(\frac{e^{-it}+e^{it}}{2}\right) + t_0\left(\frac{e^{i\tau}-e^{-it}}{2i}\right) + \frac{1}{2}\left(\frac{e^{i(t-2t_0)}+e^{-i(\tau-2t_0)}}{2}\right)$$

$$X_1 = -t\sin t - \frac{1}{2}\cos t + t_0\sin\tau + \frac{1}{2}\cos(\tau-2t_0)$$

$$\therefore X_1 = X_2 = -t\sin t - \frac{1}{2}\cos t + t_0\sin t + \frac{1}{2}\cos(\tau-2t_0)$$

$$X_3 = -\frac{(D-1)^2\,2\cos t}{D^2(D^4-1)} = -\frac{D^2+1}{D^2(D^2+1)}\,2\cos t$$

then

$$\frac{-D^2+1}{D^2(D^2+1)} = \frac{A_1}{D^2} + \frac{A_2}{D} + \frac{A_3}{D+i} + \frac{A_4}{D-i}$$

The values of the A's are found by the partial fraction process

$$A_1 = 1 \quad ; \quad A_2 = 0$$
$$A_3 = \frac{1}{i} \quad ; \quad A_4 = -\frac{1}{i}$$

$$X_3 = \left[\frac{1}{D^2+0} + \frac{\frac{1}{i}}{D+i} + \frac{-\frac{1}{i}}{D-i}\right]2\cos t$$

$$= \left[(D+0)^{-2} + \frac{1}{i}(D+i)^{-1} - \frac{1}{i}(D-i)^{-1}\right]2\cos t$$

$$= 2e^{ot}\int_{t_0}^{\tau} e^{ot}\cos t\,dt + \frac{2}{i}e^{-it}\int_{\tau_0}^{\tau} e^{it}\cos t\,dt - \frac{2}{i}e^{it}\int_{\tau_0}^{\tau} e^{-it}\cos t\,dt$$

Let $\cos t = \dfrac{e^{it}+e^{-it}}{2}$; $\sin t = \dfrac{e^{it}-e^{-it}}{2i}$

$$= 2e^{ot}\int_{\tau_0}^{\tau} e^{ot}\left(\frac{e^{it}+e^{-it}}{2}\right)dt + \frac{2}{i}e^{-it}\int_{\tau_0}^{\tau} e^{it}\left(\frac{e^{it}+e^{-it}}{2}\right)dt - \frac{2}{i}e^{it}\int_{\tau_0}^{\tau} e^{-it}\left(\frac{e^{it}+e^{-it}}{2}\right)dt$$

$$= \int_{\tau_0}^{\tau}(e^{it}+e^{-it})\,dt + \frac{1}{i}e^{-it}\int_{\tau_0}^{\tau}(e^{2it}+1)\,dt - \frac{1}{i}e^{it}\int_{\tau_0}^{\tau}(1+e^{-2it})\,dt$$

$$= \left[\frac{e^{it}}{i} - \frac{e^{-it}}{i}\right]_{\tau_0}^{\tau} + \frac{1}{i}e^{-it}\left[\left(\frac{e^{2it}}{2i} + t\right)\right]_{\tau_0}^{\tau} - \frac{1}{i}e^{it}\left[\left(t - \frac{e^{-2it}}{2i}\right)\right]_{\tau_0}^{\tau}$$

$$= \frac{e^{it}}{i} - \frac{e^{-it}}{i} - \frac{e^{it}}{2} + \frac{te^{-it}}{i} - \frac{te^{it}}{i} - \frac{e^{-it}}{2} + \frac{e^{-it_0}}{i} - \frac{e^{it_0}}{i} + \frac{e^{-i(\tau-2t_0)}}{2}$$

x

$$-\frac{t_0 e^{-it}}{i} + \frac{t_0 e^{it}}{i} + \frac{e^{i(t-2t_0)}}{2}$$

Combining

$$= \left(\frac{e^{it}-e^{-it}}{i}\right) - \left(\frac{e^{it_0}-e^{-it_0}}{i}\right) - \left(\frac{e^{it}+e^{-it}}{2}\right) - \left(\frac{te^{it}-te^{-it}}{i}\right)$$

$$+ \left(\frac{t_0 e^{it}-t_0 e^{-it}}{i}\right) + \left(\frac{e^{i(t-2t_0)}+e^{-i(t-2t_0)}}{2}\right)$$

Multrplying and dividing by 2 and factoring out t and t_0 yields

$$= 2\left(\frac{e^{it}-e^{-it}}{2i}\right) - 2\left(\frac{e^{it_0}-e^{-it_0}}{2i}\right) - \left(\frac{e^{it}+e^{-it}}{2}\right) - 2t\left(\frac{e^{it}-e^{-it}}{2i}\right)$$

$$+ 2t_0\left(\frac{e^{it}-e^{-it}}{2i}\right) + \left(\frac{e^{i(t-2t_0)}+e^{-i(t-2t_0)}}{2}\right)$$

$$= 2\sin t - 2\sin t_0 - \cos t - 2t\sin t + 2t_0 \sin t + \cos(t-2t_0)$$

$$\therefore X_3 = 2(1-t)\sin t - 2\sin t_0 - \cos t + 2t_0 \sin t + \cos(t-2t_0)$$

CHAPTER V

SUMMARY

The operator methods discussed and applied in this paper have proven to be an important simplifying and time saving device relative to solutions involving systems of linear differential equations with constant coefficients. The five theorems, of the operators, give necessary proof to this importance as time saving methods when applied to a given situation involving ordinary linear differential equations with constant coefficients. The inverse operator plays a major role in solutions employing the quotient of two polynomials in D.

To the generalization of Mr. Raymond L. Scott's paper, " Systems of Simultaneous Linear Differential Equations," which was limited to systems of the first order and whose operator polynomials were of degree 1. The characteristic determinant of the systems discussed by Mr. Scott did not exceed degree 4.

In this paper, the theorem introduced by Mr. Scott has been extended and proven that a given system may be of n equations and n variables. However we have included only operator polynomials of degree ≤ 2 and included determinant of degree 2n.

Further research may be made to apply the theorem or theorems discussed in this paper to systems of higher order and determinants of degree greater than 2n.

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